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Dynamical Phenomena near a Saddle-Focus Homoclinic Connection in a Hamiltonian System[1]

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We present main features of the orbit behavior for a Hamiltonian system in a neighborhood of homoclinic orbit to a saddle-focus equilibrium. These features includes description of hyperbolic subsets and main bifurcations when varying a value of the Hamiltonian. The proofs of results about bifurcations are given.

KEY WORDS: Hamiltonian; homoclinic; heteroclinic; hyperbolic; bifurcation.

1. INTRODUCTION

The goal of this paper is to present results about dynamical behavior, especially bifurcations, in a two-degrees-of-freedom Hamiltonian system in a neighborhood of a homoclinic orbit to saddle-focus equilibrium. Such the study allows one to comprehend deeper the structure of a Hamiltonian system in the large being in the same time more tractable technically. First results in this direction were obtained by Devaney[(1)] who carried over the impressive and unexpected results by Shilnikov[(2)] from general systems to Hamiltonian ones that required of a special (symplectic) tool. The main task in ref. 1 was to distinguish a hyperbolic subset in a neighborhood of a transverse (in a level of the saddle-focus p) homoclinic orbit to p. Namely, it was proved that a hyperbolic subset exists such that on a cross-section to orbits of this set the related Poincaré map was conjugated to the

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Bernoulli shift of 2 symbols. Since in ref. 1 the system examined in the level $H=H(p)$, a bifurcational nature of the problem was not displayed. Bifurcations in this and similar problems involving homoclinic orbits to equilibria, not to periodic orbits, appear naturally when changing the internal parameter of any Hamiltonian system—the value of its Hamiltonian. In the problem under consideration a rich bifurcational structure was indicated in ref. 3. Though proofs of these results were absent there, all principal points to carry out the proofs were presented. For an interested reader we point out that the proofs of results concerning hyperbolic behavior and related symbolic dynamics, in particular, the description of homo- and heteroclinic orbits to the saddle-focus and nearby periodic orbits have been given in ref. 4. Here we present proofs about bifurcations following the lines given in ref. 3. Namely, we show that as $c \to H(p)$ countably many times parabolic periodic orbits emerge, they break up into elliptic and hyperbolic ones, then the elliptic orbit goes through doubling, giving rise to the beginning of the doubling cascade which ends with the enlargement of the hyperbolic set (the related Bernoulli shift acquires two new states). In addition, we point out the boundary points of intervals in c where bifurcations related with changing the hyperbolic set take place.

Another problem intimately connected with the just mentioned is the structure of a Hamiltonian system near a heteroclinic connection with two saddle-foci p_1, p_2 which was studied in ref. 4. Naturally, such the connection can appear only if saddle-foci belong to the same level of the Hamiltonian. Under a perturbation p_1, p_2 generically diverse to different levels and connection breaks up. Thus, such the problem should be studied in at least one-parameter family of Hamiltonian systems. In such the setting the problem contains two parameters, the value of the Hamiltonian c and the external (governing) parameter μ. It has to expect a possibility of more complex degeneracies in such the system. For instance, we have shown in ref. 4 that, in contrast to the case of a transverse homoclinic orbit to a saddle-focus, where all nearby homoclinic orbits are transverse, here two infinite sequences $\mu_k^{(i)}$, $i=1, 2$, exist such that the system at $\mu=\mu_k^{(i)}$, has a nontransverse homoclinic orbit to p_i with the quadratic tangency.

One more important reason of the interest to such the homoclinic phenomena is a possibility to understand scenaria of appearance, the existence and the structure of localized (pulses and fronts) traveling waves and stationary patterns to parabolic gradient-like 1D PDEs. Such the solutions can be temporally stable.[5]

The next section contains the setting up and statements of the main theorems. The necessary technical assertions are given in Section 3. Section 4 is devoted to the proof of the theorem on bifurcations. Results of ref. 4 about hyperbolic behavior are essentially used in the paper.

2. SETTING UP THE PROBLEM AND MAIN RESULTS

Let (M, Ω) be a smooth (analytic or C^∞) four-dimensional symplectic manifold with symplectic form Ω. Consider a C^r-smooth Hamiltonian vector field X_H (necessary smoothness will be specified later on) with Hamiltonian H such that X_H has a singular points p of saddle-focus type. The latter means the spectrum of a linearization operator of X_H at p consists of a quadruple of eigenvalues $\pm\alpha \pm i\beta$, $\alpha\beta \neq 0$. Such the point p has two local smooth submanifolds, stable one W^s and unstable W^u, lying both in the level $H = H(p)$. This set, outside of singular points, is a smooth 3-dimensional submanifold. In particular, stable and unstable manifolds of the same or different saddle points (if they belong to the same level) generically intersect each other transversely.

Main Assumption. There is a homoclinic orbit Γ to p being transverse intersection of W^s, W^u along Γ in the level $H = H(p)$. Such the orbit is usually called the transverse homoclinic orbit to p though, strictly speaking, this intersection is not transverse in M.

A general problem is to describe the orbit behavior of nearby orbits in some neighborhood U of Γ. It is worth emphasizing that we consider only those orbits of X_H which lie entirely in U for all t. It turns out that even this problem is too hard as it will be clear from our results. Particular results in this direction are presented below.

To describe orbit behavior near the homoclinic connection we need in some notions of the symbolic dynamics (see, for instance, ref. 6). A symbolic system is constructed by means of a compact topological space called the alphabet $\mathscr{A}$, and some continuous mapping $\mathscr{T}: \mathscr{A} \times \mathscr{A} \to \{0, 1\}$ called a transition matrix. The symbolic system consists of the space Y being a set of all two-sided sequences $(..., \omega_{-1}, \omega_0, \omega_1, ...)$ with the fixed zero position (the topology is given by the Tichonov product structure) such that any two symbols ω_i, ω_{i+1} can follow one after another iff $\mathscr{T}(\omega_i, \omega_{i+1}) = 1$, and of a continuous map σ: $Y \to Y$ being a shift to the left for any symbolic sequence. This symbolic system denotes (Y, σ). For our case we use the following alphabets and transition matrices. Consider a compact countable space $\mathscr{B}$ consisting of points $\pm n^{-1}$, $n \in \mathbf{N}$ complemented with the nonseparable two-point space $\{0^+, 0^-\}$. This set has the discrete topology everywhere except for 0^+, 0^-, and neighborhoods of the point 0^+ are the sets $\{n^{-1}, n \geqslant k > 0\}$ along with the points 0^+, 0^-, the sets $\{-n^{-1}, n \geqslant k > 0\}$ along with the points 0^-, 0^+ are the neighborhoods for 0^-.

As an alphabet we take a set $\mathscr{B}$, transitions are described as follows: (i) after symbol 0^+ can follow only 0^+; (ii) after any symbol of $\mathscr{B}$, but not

0^+, can follow any symbol from $\mathscr{B}$ excepting for 0^-; (iii) only 0^- can precede 0^-. The corresponding symbolic system is denoted (Y_0, σ). Another symbolic system we use is (Y_m, σ) (Bernoulli shift), here the alphabet is $\{\pm n^{-1}, n=1,..., m\}$ and all transitions are admissible.

The orbit behavior in some neighborhood U of Γ is described via the description of orbits of the related Poincaré map on some cross-section N to Γ. This section near the trace of Γ on it is foliated by levels $H=c$ of the Hamiltonian into two-dimensional symplectic disks N_c with respect to the restriction of 2-form Ω. Thus, one obtains a one-parameter family of symplectic maps $P_c: N_c \to N_c$. The first theorem describes hyperbolic subsets existing in any level N_c. Recall that we describe only those orbits which lie in U for $t \in \mathbf{R}$.

Theorem 1. (1) At $c=0$ Poincaré map P_0 on N_0 is conjugated to the symbolic system (Y_0, σ). (2) There is $c_0 > 0$ such that for $|c| \leqslant c_0$ in the level $H=c$ an invariant hyperbolic subset exists for which the related Poincaré map is conjugated to symbolic system (Y_m, σ), where $m = n(c)$, and function $n(c)$ has the following asymptotics as $|c| \to 0$: $n(c) \sim -(\beta/2\pi\alpha) \ln |c| + \text{const}$. (3) In a segment $[-c_0, c_0]$ there is a countable set of accumulating zero disjoint intervals I_n, $n \in \{\mathbf{Z} \backslash 0\}$, such that for $c \in I_n$ the set of all orbits lying entirely in $U \cap \{H=c\}$ coincides with the hyperbolic subset of the item 2.

We call intervals of the item 3 hyperbolicity intervals. In accordance with the construction, periodic orbits of X_H correspond to periodic points of Poincaré map P, moreover, fixed points of P give periodic orbits of the field that make one round along Γ, n-periodic points of P give n-round periodic orbits of the field. In the same way the notion of n-round homoclinic orbits is introduced: these are homoclinic orbits of X_H which are homotopic to $n\Gamma$ in a thin tube near Γ. The proof of the Theorem 1 is given in ref. 4. It relies on several auxiliary assertions which are presented below.

The construction of hyperbolic subsets gives the following property of these sets. If one fixes the number $2n$ of states in the Bernoulli shift then the hyperbolic set with this number of states exists for all values of c with $|c| < c_n < c_0$. In particular, for $|c| \leqslant c_0$ there exists a hyperbolic set with 2 states. This set contains two fixed saddle points, one orientable and one nonorientable. Stable and unstable manifolds of the orientable saddle periodic orbit play an essential role in detecting boundaries of bifurcational intervals in c (see, Subsection 4.1).

It follows from the Theorem 1 that for any $n \in \mathbf{N}$ there are orbits corresponding to sequences $(..., 0^-, 0^-, a_1, a_2,..., a_n, 0^+, 0^+,...)$. These orbits are homoclinic to p and $n+1$ is their roundness.

Corollary 1. In a neighborhood of Γ there are countably many homoclinic orbits of any roundness.

Remark 1. It follows from the proof of this theorem that all these homoclinic orbits are transverse as Γ itself.

Further assertions concern with the bifurcational phenomena occurring when c varies near $H = H(p)$. Theorem 1 implies that, as $|c| \to 0$, the number of states in the related Bernoulli shift (Y_m, σ) increases, hence, bifurcations have to occur giving rise reconstructions in the orbit structure in levels $H = c$. It turns out that on the segment $[-c_0, c_0]$ in the complementary set to hyperbolicity intervals there are subintervals such that when c runs them bifurcations really take place.

Let us fix $c > 0$ to be definite, and denote (c'_{n+1}, c''_{n+1}), (c'_n, c''_n) two neighboring hyperbolic intervals, $c''_{n+1} < c'_n$.

Theorem 2. (1) In each interval (c''_{n+1}, c'_n) a subinterval J_n exists such that in J_n there are points $d_0 > d_1$ corresponding to the following bifurcations of the Poincaré map P_c: (i) at $c = d_0$ inside of rectangle N_c a parabolic fixed point appears which breaks up for $c < d_0$ into elliptic and hyperbolic fixed points, both of them persist till $c = d_1$; (ii) at $c = d_1$ the elliptic point becomes a degenerate fixed point with double multiplier -1, two-dimensional Jordan box of the linearization matrix and nonzero Lyapunov value that leads to its doubling for $c < d_1$ and appearing a period 2 elliptic periodic point, the degenerate fixed point changes into a nonorientable saddle fixed point.

The same is valid for $c < 0$.

Remark 2. The bifurcation occuring at $c = d_1$ is, in fact, the beginning of a doubling cascade leading to the formation of new Smale horseshoe constructed on two saddle fixed points, namely, the orientable saddle (with positive eigenvalues) appearing from the parabolic point after its destruction and nonorientable (Möbius) saddle having appeared from the elliptic point in the process of the first doubling. See ref. 12, where this process is discussed in more details.

3. AUXILIARY RESULTS

We use the Moser's normal form[9] to represent the local flow. Though it was found for analytic Hamiltonians, it also works in C^∞-case (Lychagin) and sufficiently smooth case.[7, 8] It is sufficient the Hamiltonian to be C^{12}, then by means of C^4-smooth symplectic transformation it can be brought into the following normal form in a symplectic frame (x_1, x_2, y_1, y_2), $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, near a saddle-focus

$$H(x_1, x_2, y_1, y_2)$$

$$= h(\xi, \eta) = \alpha\xi + \beta\eta + \cdots, \qquad \xi = x_1 y_1 + x_2 y_2, \quad \eta = x_1 y_2 - x_2 y_1 \tag{1}$$

with a polynomial h. If H C^r-smoothly depends on a parameter $\mu \in (-\mu_0, \mu_0)$ then for all $|\mu|$ small enough there are symplectic coordinates C^r-smoothly depending on μ such that H_μ has the same form (1), only h will depend on μ.

Functions ξ, η are local integrals of the flow, and equations are immediately integrated (all details of this calculations can be found in ref. 4). The orbit behavior is studied by means of related Poincaré map constructed on some cross-sections N^s, N^u to stable and unstable manifolds of p. They are foliated by levels $H = c$ into annulae N^s_c, N^u_c. For the case of homoclinic connection this map is a superposition of two maps, local one near p and global one, near a global piece of Γ. Using (1) and introducing a function $\xi = a_c(\eta) = \alpha^{-1}(c - \beta\eta + \cdots)$ being a unique solution of the equation $h(\xi, \eta) = c$ with respect to ξ in some neighborhood of p, we obtain the following representation of the local map T_c[1, 3, 4]

$$\varphi = \theta + B_c(\eta) (\text{mod } 2\pi), \qquad \eta = \eta \tag{2}$$

where (θ, η) and (φ, η) are local coordinates on the sections, if $c = 0$ the segments $\eta = 0$ correspond to the traces of stable and unstable manifolds, respectively. The local map obtained is symplectic, it is discontinuous along the circle $\eta = 0$ for $c = 0$ and it is smooth for $c \neq 0$. The properties of function $B_c(\eta)$ are described below.

Global map S_c is defined in some neighborhoods of traces of Γ on the sections, and is given in coordinates (θ, η, c) on the sections N^s and (φ, η, c) on N^u as

$$\theta_1 = f(\varphi, \eta, c) = f_c(\varphi, \eta), \qquad \eta_1 = g(\varphi, \eta, c) = g_c(\varphi, \eta) \tag{3}$$

with $D(f, g)/D(\varphi, \eta) \equiv 1$ (symplecticity), $f(0, 0, 0) = g(0, 0, 0) = 0$ (as the trace of Γ on N^u transforms to the trace of Γ on N^s), $(\partial g/\partial\varphi)(0, 0, 0) \neq 0$ (transversality condition of W^s and W^u).

All further considerations are carried out in some neighborhoods of points $\Gamma \cap N^s$, $\Gamma \cap N^u$. These neighborhoods Π^s, Π^s are determined by inequalities

$$\Pi^s = \{(\theta, \eta, c) \mid |\theta| \leqslant \delta, |\eta| \leqslant \varepsilon, |c| \leqslant c_0\}$$

$$\Pi^u = \{(\varphi, \eta, c) \mid |\varphi| \leqslant \delta, |\eta| \leqslant \varepsilon, |c| \leqslant c_0\}$$

for ε, δ, c_0 small enough. The levels $V_c = \{H = c\}$ are invariant sets, so we obtain a family of Poincaré maps depending on parameter c, given on rectangles $\Pi^s_c = \Pi^s \cap V$, $\Pi^u_c = \Pi^u \cap V_c$.

The properties of the local map T are formulated in the next lemmas, their proofs are given in ref. 4. Here and later on we denote $O_k(x)$ a function that is given on a neighborhood of $x = 0$ and such that $O_k(x)/x^k$ is bounded as $x \to 0$, $o(x)$ means that $o(x)/x \to 0$ as $x \to 0$, and $O(x)$ denotes a function which tends to zero as $x \to 0$.

Lemma 1. For $|c|$, $|\eta|$ small enough the following holds

$$B_0'(\eta) = \frac{\beta/\alpha + O(\eta)}{\eta}, \qquad |B_0'| \leqslant \frac{\beta}{2\alpha\, |\eta|} \tag{4}$$

$$B_c'(\eta) = [L(c, \eta) + a_c'' R(c, \eta) + O_2(c, \eta)]/(\eta^2 + a_c^2(\eta)) \tag{5}$$

with $L(c, \eta) = \alpha^{-3}(\alpha^2 - \beta^2)\, c + \beta(\beta^2 + \alpha^2)\, \eta$, $R(c, \eta) = (\eta^2 + a_c^2(\eta)) \ln(\eta^2 + a_c^2(\eta))$;

$$B_c''(\eta) = [q_2(c, \eta) + O_3(c, \eta)]/(\eta^2 + a_c^2(\eta))^2 \tag{6}$$

with a quadratic form $q_2(c, \eta) = \alpha^{-2}\sigma(3 - \sigma^2)\, c^2 + 2\alpha^{-1}(\sigma^4 - 1)\, c\eta - \sigma(1 + \sigma^2)^2\, \eta^2$, $\sigma = \beta/\alpha$, having positive discriminant $\Delta = \alpha^{-2}(\sigma^2 + 1)^3$.

To formulate next lemma let us consider a standard covering of the annulus N^u_c. It is a strip on the plane (φ, η), where φ is considered as affine coordinate, $|\eta| \leqslant \varepsilon$. If $\theta = u(\eta)$ is a function given for $|\eta| \leqslant \varepsilon$, then the image of its graph w.r.t. T_c is a curve in the strip (φ, η), being graph of a function (see (2))

$$\varphi = u(\eta) + B_c(\eta) \tag{7}$$

Lemma 2. There are positive ε, c_0 small enough such that in the strip (φ, η), $|\eta| \leqslant \varepsilon$, the image under $T(c)$ of the graph of a C^2-function $\theta = u(\eta)$, $|u(\eta)| \leqslant \delta$, $|u'(\eta)| \leqslant d_1$, $|u''(\eta)| \leqslant d_2$, is

1. for $c=0$ the graph of a function $\varphi(\eta)$ being C^2-smooth everywhere on $|\eta| \leqslant \varepsilon$ except for the point $\eta=0$ where it has a logarithmic singularity, derivative of $\varphi(\eta)$ satisfies the estimate $|\varphi'(\eta)| \geqslant \beta/2\alpha\,|\eta|$;

2. for $c \neq 0$, $|c| \leqslant c_0$, c_0 small enough, the graph of a C^2-smooth function $\varphi(\eta)$ such that

(i) $\varphi'(\eta)$ is a monotone function with a unique zero at a minimum point η_c, $\eta_c = ((\beta^2-\alpha^2)/\beta(\beta^2+\alpha^2))\,c + o(c)$, $o(c) \to 0$ as $|c| \to 0$;

(ii) the value $\varphi(\eta_c)$ tends to $-\infty$ when $|c| \to 0$, moreover, the following representation is valid: $\varphi(\eta_c) = (\beta/\alpha)\ln|c| + E(c)$ with a bounded function $E(c)$, and $(d/dc)\,\varphi(\eta_c) = (\beta/\alpha + O(c))/c$.

The following lemma allows one to distinguish the region of hyperbolicity and a band on Π_c^s where the creation of parabolic fixed points occurs.

Lemma 3. For any $K>0$ there exists $\gamma>0$ such that for all $|c| \leqslant c_0$ there is a region on the segment $|\eta| \leqslant \varepsilon$, where the estimate $|B_c'(\eta)| \geqslant K$ holds. Furthermore,

if $c=0$, then this region coincides with the segment $|\eta| \leqslant \varepsilon$;

if $c \neq 0$, then this region consists of two segments given with inequalities $\eta_c + \gamma c^2 \leqslant \eta \leqslant \varepsilon$ and $-\varepsilon \leqslant \eta \leqslant \eta_c - \gamma c^2$.

Next lemma is used for proofs that tangency is quadratic if stable and unstable manifolds of some periodic orbit in a neighborhood of Γ are tangent.

Lemma 4. Consider a family of smooth C^2-functions of the form $\varphi = v(\eta)$, $|\eta| \leqslant \varepsilon$ with C^2-norms bounded with some constant D. Then there is a positive c_1 small enough such that for all c, $|c| \leqslant c_1$, graphs of any function $\varphi(\eta)$ from Lemma 2 and of $v(\eta)$ are quadratically tangent if they have a tangency point.

Now we are able to describe the domain of the map T and its restrictions T_c (see (2)). Let us denote $\eta = \lambda_{\pm}(\xi)$ two branches of the inverse function for $\xi = B_c(\eta)$. Fix $\varepsilon>0$, $\delta>0$ such that conclusions of preceding Lemmas 1–4 would hold.

1. $c=0$. Then for any θ, $|\theta| \leqslant \delta$, curves $\varphi = \theta + B_0(\eta)$ (here θ is a parameter marking the curve) monotonically decrease for $\eta<0$ and increase for $\eta>0$. Since $B_0(\eta) \to -\infty$ as $|\eta| \to 0$, then graphs of inverse functions $\eta = \lambda_+(\varphi-\theta)$ and $\eta = \lambda_-(\varphi-\theta)$ being projected into the annulus N_0^u are the curves which go round the annulus infinitely many times

approaching the circle $\eta=0$ as $\varphi\to-\infty$. Take two such the curves with $\theta=\pm\delta$. Then, beginning from some $n_0>0$ these curves will intersect all segments $\varphi=\delta-2\pi\,|n|$, $n\geqslant n_0$, $|\eta|\leqslant\varepsilon$. Thus, we obtain infinitely many strips

$$\begin{aligned}
\sigma_n^u&=\{(\varphi,\eta)\mid|\varphi|\leqslant\delta,\lambda_+(\varphi-\delta+2\pi n)\leqslant\eta\leqslant\lambda_+(\varphi+\delta+2\pi n)\},\\
&\qquad\text{if}\quad n>0\qquad(\text{i.e., }\eta>0)\\
\sigma_n^u&=\{(\varphi,\eta)\mid|\varphi|\leqslant\delta,\lambda_-(\varphi+\delta-2\pi n)\leqslant\eta\leqslant\lambda_-(\varphi-\delta-2\pi n)\},\\
&\qquad\text{if}\quad n<0\qquad(\text{i.e., }\eta<0)
\end{aligned}\tag{8}$$

being the domain of T_0.

2. $c\neq0$. Lemma 2 implies that curves $\varphi=\delta+B_c(\eta)$ for all $|n|\geqslant n_0$ intersect segments $\varphi=\delta-2\pi\,|n|$, $n_0\leqslant n\leqslant n_1(c)$, $|\eta|\leqslant\varepsilon$, where $n_1(c)$ is equal to that maximal n such that $\delta+B_c(\eta)+2\pi n<-\delta$. For these n equation $\varphi=\theta+B_c(\eta)$ can be solved for $\theta=\pm\delta$ giving for $n>0$ lower (at $\theta=\delta$) and upper (for $\theta=-\delta$) boundaries of the strip σ_n^u. If $n<0$ then lower and upper boundaries change places. The difference $n_1(c)-n_0$ gives the upper estimate for the number of strips. In fact, in order to get hyperbolicity for the related invariant sets, we need to throw away some finite number of initial sets. In fact, hyperbolicity can be proved for intervals $|\eta-\eta_c|\geqslant\gamma c^2$ where $|B_c'|>K$. One can be shown (see ref. 4) that asymptotically

$$n(c)\sim-\frac{\beta}{2\pi\alpha}\ln|c|+\text{const}\tag{9}$$

We have constructed strips σ_n^u making up the range of the local map T. The domain of this map, σ_n^s, are preimages of σ_n^u, and they are determined with inequalities similar to (8) with $\lambda_\pm(\pm\delta-\theta-2\pi\,|n|)$.

4. PROOF OF THEOREM 2

We look for fixed points of the Poincaré map as follows. For the global map S_0 the inequality $(\partial g_0/\partial\varphi)(0,0)\neq0$ holds, $f_0(0,0)=0$, $g_0(0,0)=0$, then for some positive small c_0 and $|c|<c_0$ the second equation in (3) can be solved w.r.t. φ, therefore (3) can be rewritten in the "cross" form

$$\theta_1=Q_c(\eta,\eta_1)=-\frac{\partial F_c}{\partial\eta_1},\qquad\varphi=R_c(\eta,\eta_1)=\frac{\partial F_c}{\partial\eta}\tag{10}$$

for $|\eta|$, $|\eta_1|$, $|\varphi|$, $|\theta|$ small enough, here F_c is a generating function of the symplectic map. Using (2) the equations for finding fixed points take the form

$$Q_c(\eta, \eta) = \theta (\text{mod } 2\pi), \qquad R_c(\eta, \eta) = \theta + B_c(\eta) (\text{mod } 2\pi)$$

Eliminating θ from these equations we come to the equation

$$r_c(\eta) = B_c(\eta) (\text{mod } 2\pi), \qquad \text{with} \quad r_c(\eta) = R_c(\eta, \eta) - Q_c(\eta, \eta) = \frac{\partial}{\partial \eta} F_c(\eta, \eta) \tag{11}$$

It is easily seen that the function $\psi = r_c(\eta)$ in l.h.s of this equation is a smooth function in η, c, which vanishes at $\eta = 0$, $c = 0$. Considering it as a family of smooth functions of η depending smoothly on a parameter c we get that the functions of this family are C^3-close to that which corresponds to $c = 0$. The graph of this latter function contains the point $(0, 0)$, therefore graphs of all functions of the family lie in the band $|\psi| \leqslant \delta$ for ε small enough.

On the other hand, due to Lemmas 2, 3 the function $B_c(\eta)$ on the segment $[-\varepsilon, \varepsilon]$ has a unique minimum that monotonically tends to $-\infty$ as $|c| \to 0$. Considering its graph in the strip $|\eta| \leqslant \varepsilon$, $-\pi \leqslant \psi \leqslant \pi$ one obtains that it consists of finitely many branches with their range the segment $[-\pi, \pi]$ and a middle part in the form of a parabola-like sharp tongue that stretches monotonically till $\psi = -\pi$ when decreasing $|c|$. It implies that these two graphs always have finitely many points of the transverse intersection inside of the band $|\psi| \leqslant \delta$, these intersection points correspond to fixed points of the hyperbolic set (they also correspond to the fixed points of the Bernoulli shift), and at any passage of the tongue through the band one obtains a point of tangency of these graphs. So, we have countably many such values of c when $|c| \to 0$. That the tangencies are quadratic follows from the representation (6) implying $|B_c''(\eta)| \to \infty$ in the region $|\eta - \eta_c| \leqslant \gamma c^2$ where tangencies occur.

Remark 3. It should be emphasized that the considerations presented essentially use the fact that the local map T_c is defined and its properties are known (in fact, those of the function B_c) in the neighborhood of the trace of whole stable manifold ($\eta = 0$ here), not only in a neighborhood of the trace of the homoclinic orbit Γ.

In order to connect the intersection points of the graphs and fixed points of the map let us apply the idea from ref. 10 which connects fixed points of an area preserving map and critical points of its generating function.

Lemma 5. Let $S: (x, y) \to (x_1, y_1) = (f(x, y), g(x, y))$ be a symplectic map and suppose $g_x \neq 0$ in some simply connected region G such that the map can be written in the cross form $x = P(y, y_1)$, $x_1 = Q(y, y_1)$ with a generating function $F(y, y_1)$, that is, $P(y, y_1) = F_y$, $Q(y, y_1) = -F_{y_1}$, and $F(y, y_1)$ is defined in some simply connected region D, where $F_{yy_1} \neq 0$. Then if (x_*, y_*) is any isolated fixed point of S in G then y_* is an isolated critical point of the function $f(y) = F(y, y)$. Conversely, if y_* is a critical point of this function such that the point (x_*, y_*), $x_* = F_y(y_*, y_*)$ belongs to G then (x_*, y_*) is the fixed point of S. Moreover, nondegenerate critical points of the generating function correspond to hyperbolic and elliptic points of the map in dependence of the sign of the second derivative, and vice versa.

For our case the Poincaré map takes the form

$$\theta_1 = f_c(\theta + B_c(\eta) + 2\pi k, \eta), \qquad \eta_1 = g_c(\theta + B_c(\eta) + 2\pi k, \eta) \tag{12}$$

As we have already known (see Section 3), the domain of the local map T_c consists of either countably many strips for $c = 0$, or of the finite number of strips (always exist for $c \neq 0$) and, in addition for some c, of a middle connected part where B_c' can vanish ("dangerous" zone). For these latter c there are a positive integer k and values of (θ, η) such that the value of the first argument in f_c, g_c in (12) belongs to the interval $(-\delta, \delta)$. A generating function $\hat{F}_c(\eta, \eta_1)$ of the map is $F_c(\eta, \eta_1) - 2\pi k\eta - \int^\eta B_c(s)\, ds$, therefore,

$$\frac{\partial \hat{F}_c}{\partial \eta} = \frac{\partial F_c}{\partial \eta} - B_c(\eta) - 2\pi k, \qquad \frac{\partial \hat{F}_c}{\partial \eta_1} = \frac{\partial F_c}{\partial \eta_1}$$

So, using (10) we get

$$\theta = R_c(\eta, \eta_1) - B_c(\eta) - 2\pi k = \frac{\partial \hat{F}_c}{\partial \eta}, \qquad \theta_1 = Q_c(\eta, \eta_1) = -\frac{\partial \hat{F}_c}{\partial \eta_1} \tag{13}$$

and the equation for searching for critical points is

$$f_c'(\eta) = \left(\frac{\partial \hat{F}_c}{\partial \eta} + \frac{\partial \hat{F}_c}{\partial \eta_1}\right)\bigg|_{y = y_1} = R_c(\eta, \eta) - B_c'(\eta) - 2\pi k - Q_c(\eta, \eta) = 0$$

that is, it precisely coincides with (11). To determine the types of appearing fixed points of the map we use the following assertion.[10]

Lemma 6 (Parabolicity Conditions). Let, under the conditions of the preceding Lemma, the critical point be simplest degenerate, i.e., $f''(y_*) = 0$ but $f'''(y_*) \neq 0$. Then, the corresponding fixed point of the map is parabolic, that is, it has 1 as a double multiplier, Jordan form of the related linearization matrix is 2-dimensional box, and the related coefficient (see below) in the normal form of the second order at this point does not vanish. If, in addition, the family of the functions f depends smoothly on a parameter c, and at $c=0$ a simplest degenerate critical point exists at which $f_y'''(y_*,0)\neq 0$ and $(\partial^2/\partial c\,\partial y)\,f(y_*,0)\neq 0$, then, when passing through $c=0$ the following bifurcation of the map occurs: on the one side of $c=0$ the related map has not fixed points near (x_*,y_*), but on the other side there are two fixed points, elliptic and hyperbolic ones.

The related normal form of the second order to which any area preserving map near its parabolic point can be transformed is the following

$$x_1=x+y+Ax^2+\cdots,\qquad y_1=y+Ax^2+\cdots$$

Parabolicity condition (i.e., not more higher degeneracy) is $A\neq 0$. Two-dimensionality of Jordan box follows from the inequality $g_x\neq 0$.

This Lemma implies that a fixed point will be a parabolic if it corresponds to a simplest degenerate critical point of generating function. The Lemma works in our case. Indeed, let us calculate $f_c''(\eta_*)$. As is easily seen, in notations of Lemma 5, this quantity is equal to the value of function $F_{yy}+2F_{yy_1}+F_{y_1y_1}$ evaluated at the point $y=y_1=y_*$. Since the trace of Jacobi matrix is $f_x+g_y=P_{y_1}^{-1}(Q_{y_1}-P_y)$, then $f_x+g_y-2=P_{y_1}^{-1}(Q_{y_1}-P_y-2P_{y_1})=-(F_{y_1y_1}+F_{yy}+2F_{yy_1})/F_{yy_1}$, the numerator of this fraction at the point $y=y_1=y_*$ is equal to $f''(y_*)$, this implies that the trace is equal to 2 if and only if $f_c''(\eta_*)=0$. This calculation shows that vanishing this quantity is equivalent to tangency of curves in (11), and their transversality means that the related critical point is nondegenerate. Since $|B_c'(\eta)|$ is large enough in the region $|\eta-\eta_c|\geqslant\gamma c^2$, then all intersection points of two graphs over this region are transverse. Thus, we have got countably many points with the double multiplier 1, one needs to verify that they are parabolic.

The condition $f'''(y_*)\neq 0$ reads in our case as nonvanishing the quantity

$$f_c'''(\eta_*)=-B_c''(\eta)+\frac{\partial^2 R_c}{\partial\eta^2}(\eta,\eta_1)+2\,\frac{\partial^2 R_c}{\partial\eta\,\partial\eta_1}(\eta,\eta_1)$$
$$+\frac{\partial^2 R_c}{\partial\eta_1^2}(\eta,\eta_1)-\frac{\partial^2 Q_c}{\partial\eta^2}(\eta,\eta_1)-2\,\frac{\partial^2 Q_c}{\partial\eta\,\partial\eta_1}(\eta,\eta_1)-\frac{\partial^2 Q_c}{\partial\eta_1^2}(\eta,\eta_1)$$

where one should set $\eta = \eta_1 = \eta_*$ in the function at the r.h.s.. Similarly, the function $(\partial^2/\partial c\, \partial y)\, f$ takes the form

$$\frac{\partial^2}{\partial c\, \partial y} f = -\frac{\partial B_c'(\eta)}{\partial c} - \frac{\partial Q_c}{\partial c} + \frac{\partial R_c}{\partial c}$$

The properties of the function $B_c(\eta)$ (Lemma 1), namely, $B_c''(\eta_*) \to \infty$, and $|(\partial/\partial c)\, B_c(\eta_c)| \to \infty$ prove Lemma 6. From this we get a countable set of $c = d_0(n)$ in each semi-interval $c > 0$ and $c < 0$.

The points $c = d_1(n)$ are obtained from another statement connecting the presence of a fixed point with the double multiplier -1 with some properties of generating function of the map under consideration.

Lemma 7. Let, under the conditions of Lemma 5, a critical point y_* of the function $F(y, y)$ be such that the function $F_{yy} - 2F_{yy_1} + F_{y_1 y_1}$ evaluated at the point $y = y_1 = y_*$ is equal to zero. Then the related fixed point (x_*, y_*) has double multiplier -1 with two-dimensional Jordan box. If, in addition, F depends on a parameter c and quantities similar to those in Lemma 6 do not vanish, then for $|c|$ small enough the following bifurcation does occur: in the space (x, y, c) near the point $(0, 0, 0)$ there is a smooth curve $(x(c), y(c), c)$ through $(0, 0, 0)$ which consists of fixed points of the map. The fixed points are elliptic for $c < 0$ and they are hyperbolic for $c > 0$, or vice versa. Furthermore, from this family of fixed points a family of period 2 points branches at $c = 0$. This latter family exists only on one side of $c = 0$. In dependence on the sign of some coefficient in the normal form of the second order the bearing family consists of either elliptic period 2 points (then it exist for those c where the main family consists of hyperbolic points) or, for opposite sign of the coefficient, it consists of hyperbolic period 2 points (then it exists for those c where the main family consists of elliptic points).

Proof. Let us calculate the trace of linearization matrix $D(f, g)/D(x, y)$. It is easily verified that $f_x + g_y = P_{y_1}^{-1}(Q_{y_1} - P_y)$. Therefore one gets $\sigma = f_x + g_y + 2 = P_{y_1}^{-1}(Q_{y_1} - P_y + 2P_{y_1}) = -(F_{y_1 y_1} + F_{yy} - 2F_{yy_1})/F_{yy_1}$, it implies that the trace is equal to -2 iff the numerator of this expression vanishes. Two-dimensionality of the Jordan box follows, as before, from the inequality $g_x \neq 0$.

In our case the numerator is equal to $-B_c'(\eta) + (\partial R_c/\partial \eta) + 2(\partial Q_c/\partial \eta) - (\partial Q_c/\partial \eta_1)$. So, as above, since fixed points appear in the region $|\eta - \eta_c| \leqslant \gamma c^2$, we conclude that at the bearing elliptic point the quantity $f_x + g_y$ decreases monotonically, when $|c|$ decreases, reaching the value -2. It follows from this lemma the existence of the fixed point with double eigenvalue -1, two-dimensional Jordan box for linearized map and nonzero Lyapunov value, that leads to the doubling bifurcation for the further varying c. The Theorem 2 is proved.

4.1. Bifurcational Intervals Revisited

Bifurcational intervals, inside of which bifurcations described by Theorem 2 occur, can be characterized in more details. Namely, boundary points of these intervals can be explicitly pointed out. To this end, let us enumerate strips lying outside of the region $D_c = \{|\eta - \eta_c| \leqslant \gamma c^2\}$ (see Lemma 3) in such a way that their numeration begins with 1 (for the upper strip) and -1 (for lower strip). Consider, for definiteness, the case when $(\partial g/\partial \varphi)(0, 0, 0) > 0$ in (3). We distinguish a region in $\Pi^s(c)$ bounded with segments of stable and unstable manifolds of the orientable saddle fixed point O lying in the strip σ_1^s and corresponding to the sequence $(\ldots, 1, 1, \ldots)$. Another saddle fixed point N corresponding to $(\ldots, -1, -1, \ldots)$ is nonorientable (Möbius's one). Stable manifold of the point O intersects transversely unstable manifold of N, giving a heteroclinic point q_1 (in the upper strip, and, unstable manifold of O transversely intersects stable manifold of N giving a heteroclinic point q_2 (in the lower strip) (see Fig. 1). Let us construct a curvilinear rectangle R_c in Π_c^s, whose boundaries are: the upper one is the stable manifold of the point O, the lower one is the preimage under P_c of a local piece through q_1 of the stable manifold of point O (this preimage is a smooth curve in σ^s_{-1} lying beneath the stable manifold of the point N, due to nonorientability of N, and intersecting $W^u(O)$); from the left it is bounded with unstable manifold of O, and from the right—with the image under P_c of that local piece through q_2 of the unstable manifold of O which belongs to σ^s_{-1} (this image is a smooth curve that lies to the right of the unstable manifold of N, due to nonorientability of N, and intersecting $W^s(O)$).

It is readily seen, due to the construction, that the region constructed is invariant in the sense that image (and preimage) of any strip $\sigma_c^s(j)\ R_c$ $|j| \leqslant n(c)$, belongs to this region, but points from the strip $D_c \cap R_c$ can be transformed outside of R_c. When c belongs to a hyperbolic interval, then $P_c(D_c)$ is situated out of R_c and orbits lying entirely in U cut $\Pi^s(c)$ only in strips. When $|c|$ decreases, $T_c(D_c)$ moves monotonically (see Lemma 2) around the annulus N_c^u and countably many times passes through $\Pi^u(c)$, therefore $P_c(D_c)$ monotonically passes through R_c. The first value of c, when their intersection is not empty, corresponds exactly to the first quadratic (Lemma 4) tangency point of the stable manifold of O and of the image of that piece of the unstable manifold of O (on the left side of

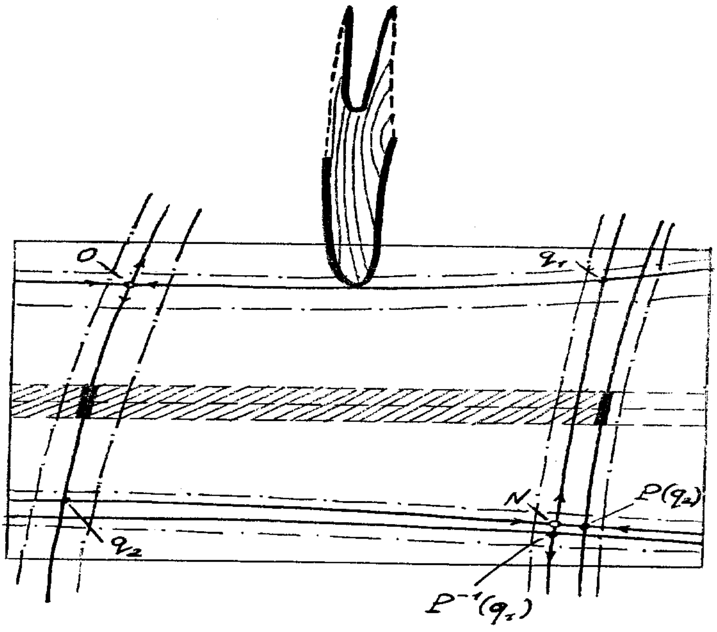

Fig. 1. The first tangency of stable and unstable manifolds of the point O. The dashed region is the dangerous one, it is determined by the inequalities $|\eta-\eta_c|\leqslant\gamma c^2$. Only two extreme strips are plotted.

the boundary of R_c) which belongs to D_c (see Fig. 1 and ref. 13 for the explanation what the first tangency point means). Before this value of c no orbits of P_c exist which begin in D_c and hit R_c one more time. After that, bifurcations take place related with formation of multi period elliptic points in the first strip in a neighborhood of the tangency point.[11] In fact, bifurcational structure, when $P_c(D_c)$ passes through R_c, is very complicated, in particular, Newhouse phenomena are expected here. Moreover, when c decreases further will be tangent with stable manifold of the orientable saddle fixed point in the strip $\sigma_2^s(c)$, etc. The last point of the bifurcational interval under consideration is that c when the image under P_c of the right boundary of R_cD_c (see above) appears to be tangent to lower boundary of R_c. It is a so-called last tangency point.[14] A neighborhood of this point can be constructed where dynamics is hyperbolic everywhere except the tangency point itself.[14] After that, when $|c|$ further decreases, hyperbolic structure of that larger invariant set of orbits restores and the whole invariant set of orbits lying entirely in N_c^s acquires two new states in the Bernoulli shift.

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